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# O-MINIMAL TOPOLOGY

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Let $R$ be a real closed field and assume an o-minimal structure over $R$. Between the last century, topology of real manifolds ($\mathbf{R}$-manifolds) was vastly and profoundly investigated. We consider manifolds over $R$ with the o-minimal structure (called definable manifolds). I will explain whether known important results on $\mathbf{R}$-manifolds hold and what properties definable manifolds have but $\mathbf{R}$-manifolds do not necessarily have.

There are three kinds of $\mathbf{R}$-manifolds: $C^0$ $\mathbf{R}$-manifolds, PL $\mathbf{R}$-manifolds and $C^r$ $\mathbf{R}$-manifolds ($r = 1, ..., \omega$). A $*$ $\mathbf{R}$-manifold ($* = C^0$, PL or $C^r$) is defined by a local $*$ coordinate system, and a PL homeomorphism between polyhedra means a homeomorphism which is linear on each simplex of some simplicial decomposition of the domain of definition. It is easy to imbed a PL $\mathbf{R}$-manifold into a Euclidean space by a PL imbedding so that the image is a polyhedron. Hence we regard a PL $\mathbf{R}$-manifold as a polyhedron in some Euclidean space. Since there is not a large difference between topology of $C^1$ $\mathbf{R}$-manifolds and topology of $C^r$ $\mathbf{R}$-manifolds ($r = 2, ...$), we consider $C^1$ $\mathbf{R}$-manifolds only.

A non-compact $C^0$ PL or $C^1$ $\mathbf{R}$-manifold is called *compactifiable* if it is homeomorphic, PL homeomorphic or $C^1$ diffeomorphic, respectively, to the interior of a compact $C^0$, PL or $C^1$, respectively, $\mathbf{R}$-manifold with boundary. The simplest example of a non-compactifiable connected $\mathbf{R}$-manifold is $\mathbf{R}^2 - \mathbf{Z} \times \{0\}$. Note only that there are non-compactifiable contractible $C^0$, PL or $C^1$ $\mathbf{R}$-manifolds and, even if compactifiable, compactifications are not necessarily unique up to homeomorphisms, PL homeomorphisms or $C^1$ diffeomorphisms, respectively.

## 1. $C^1$ $\mathbf{R}$-MANIFOLDS AND PL $\mathbf{R}$-MANIFOLDS

In the following sense we can regard a $C^1$ $\mathbf{R}$-manifold as PL $\mathbf{R}$-manifold.

**Cairns-Whitehead Theorem** (see [5]). *Given a $C^1$ $\mathbf{R}$-manifold $M$, there exist a PL $\mathbf{R}$-manifold $M^{PL}$, a simplicial decomposition $K$ of $M^{PL}$ and a homeomorphism $\pi : M^{PL} \to M$ such that $\pi|_\sigma : \sigma \to \pi(\sigma)$ is a $C^1$ diffeomorphism for each $\sigma \in K$. Such $M^{PL}$ is unique, i.e. if there is another $M_1^{PL}$ of the same properties then $M^{PL}$ and $M_1^{PL}$ are PL homeomorphic.*

Hence the correspondence $M \to M^{PL}$ induces a well-defined map $\Pi$ from the $C^1$ diffeomorphism classes of $C^1$ $\mathbf{R}$-manifolds to the PL homeomorphism classes of PL $\mathbf{R}$-manifolds. We call $M^{PL}$ a $C^1$ triangulation of $M$. Then a natural questions is whether $\Pi$ is bijective, which was posed by Thom. Namely, is any PL $\mathbf{R}$-manifold a $C^1$ triangulation of some $C^1$ $\mathbf{R}$-manifold? If $C^1$ triangulations of two $C^1$ $\mathbf{R}$-manifolds are PL homeomorphic, are the $C^1$ $\mathbf{R}$-manifolds $C^1$ diffeomorphic? Milnor, Munkres,

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Hirsch et al. studied the problems and obtained many results. For example, a PL $\mathbf{R}$-manifold of dimension $\leq 7$ and a contractible PL $\mathbf{R}$-manifold are $C^1$ triangulations of some $C^1$ $\mathbf{R}$-manifolds; there are multi $C^1$ $\mathbf{R}$-manifolds whose $C^1$ triangulations are PL homeomorphic to a PL $\mathbf{R}$-sphere (the boundary of a simplex) of dimension $\geq 7$ (we call such $C^1$ $\mathbf{R}$-manifolds exotic spheres); the restriction of $\Pi$ to manifolds of dimension $\leq 3$ is bijective (we say that a PL $\mathbf{R}$-manifold of dimension $\leq 3$ admits a unique $C^1$ $\mathbf{R}$-manifold structure).

## 2. PL $\mathbf{R}$-MANIFOLDS AND $C^0$ $\mathbf{R}$-MANIFOLDS

A $C^0$ $\mathbf{R}$-manifold of dimension $\leq 3$ admits a unique PL $\mathbf{R}$-manifold structure (i.e. a $C^0$ $\mathbf{R}$-manifold of dimension $\leq 3$ is homeomorphic to some PL $\mathbf{R}$-manifold and such PL $\mathbf{R}$-manifolds are PL homeomorphic each other). However the relation of $C^0$ $\mathbf{R}$-manifolds and PL $\mathbf{R}$-manifolds of dimension 4 is very complicated. On the other hand, the relation of manifolds of dimension $\geq 5$ is clarified by Kirby and Siebenmann.

**Kirby-Siebenmann Theorem** (see [3]). *Given a compact $C^0$ $\mathbf{R}$-manifold $M$ of dimension $\geq 5$, there is a well-defined obstruction $\tau(M)$ in $H^4(M, \mathbf{Z}_2)$ such that $M$ admits a PL $\mathbf{R}$-manifold structure if and only if $\tau(M) = 0$. Given a compact PL $\mathbf{R}$-manifolds $M$ of dimension $\geq 5$, there is one-to-one correspondence from $H^3(M_1, \mathbf{Z}_2)$ to isotopy classes of PL $\mathbf{R}$-manifold structures on $C^0$ $\mathbf{R}$-manifold $M$.*

Consequently, there exists a compact $C^0$ $\mathbf{R}$-manifold $M$ of dimension $\geq 5$ which does not admit a PL $\mathbf{R}$-manifold structure; there exist compact PL $\mathbf{R}$-manifolds $M_1$ and $M_2$ of dimension $\geq 5$ which are homeomorphic but not PL homeomorphic; a compact $C^0$ $\mathbf{R}$-manifold $M$ of dimension $\geq 5$ admits a PL $\mathbf{R}$-manifold structure if $H^4(M, \mathbf{Z}_2) = 0$; compact PL $\mathbf{R}$-manifolds $M_1$ and $M_2$ of dimension $\geq 5$ are PL homeomorphic if they are homeomorphic and $H^3(M_1, \mathbf{Z}_2) = 0$.

## 3. O-MINIMAL STRUCTURES OVER A REAL CLOSED FIELD

An *o-minimal structure* over $R$ is a sequence $\{S_n : n \in \mathbf{N}\}$ such that for each $n \in \mathbf{N}$,
(i) $S_n$ is a boolean algebra of subsets of $R^n$,
(ii) if $X \in S_n$, then $R \times X$ and $X \times R$ are elements of $S_{n+1}$,
(iii) every algebraic set in $R^n$ is an element of $S_n$,
(iv) if $X \in S_{n+1}$, then the image of $X$ under the projection of $R^{n+1}$ onto the first $n$ coordinates is an element of $S_n$, and
(v) an element of $S_1$ is a finite union of points and open intervals $(a, b) = \{x \in R : a < x < b\}$ $(a, b \in R \cup \{\pm\infty\})$.

An element of $S_n$ is called *definable*, and a map between definable sets is called *definable* if its graph is definable. We call a definable set in $R^n$ *compact* if it is closed and bounded in $R^n$. There are two fundaments on topology of definable sets.

**Triangulation Theorem**. *A definable set is definably homeomorphic to a finite union of open simplexes. A compact definable set is definably homeomorphic to a finite union of simplexes.*

**Uniqueness Theorem**. *Two definable polyhedra are definably PL homeomorphic if they are definably homeomorphic.*

We naturally define a definable $C^0$, PL or $C^1$ manifold. Then an immediate corollary is

**Corollary**. *A compact definable $C^0$ manifold is definably homeomorphic to a compact definable PL manifold, and a non-compact definable $C^0$ manifold is definably homeomorphic to the interior of a compact definable PL manifold with boundary, i.e., a non-compact definable $C^0$ or PL manifold is compactifiable. Moreover these compact definable PL manifolds possibly with boundary are unique up to PL homeomorphisms.*

Thus there is no difference between definable $C^0$ manifolds, definable PL manifolds and their compactifications. This is the difference between $\mathbf{R}$-manifolds and definable manifolds.

The triangulation theorem is proved in the same way as triangulations of semianalytic sets by Łojasiewicz [4] (see [7] and [9]). However the proof of the uniqueness theorem is very complicated [8]. It requires knowledge of PL topology, stratification theory and model theory.

## 4. COMPACT DEFINABLE PL MANIFOLDS POSSIBLY WITH BOUNDARY

It is not easy to study definable $C^0$ manifolds directly. However, by the above corollary it suffices to consider compact definable PL manifolds possibly with boundary, which are easy to treat as follows.

We naturally define a $\mathbf{Q}$-polyhedron and a PL $\mathbf{Q}$-manifold in $\mathbf{Q}^n$. Note that a compact $\mathbf{Q}$-polyhedron is a finite union of $\mathbf{Q}$-simplexes but an $R$-polyhedron closed and bounded in $R^n$ is not necessarily a finite union of $R$-simplexes if $R$ is non-Archimedean. For a $\mathbf{Q}$-simplex $\sigma$ in $\mathbf{Q}^n$, let $\sigma_R$ denote the simplex in $R^n$ spanned by the vertices of $\sigma$. For a compact $\mathbf{Q}$-polyhedron $X$ in $\mathbf{Q}^n$, we define $X_R$ to be $\cup_{\sigma\in K}\sigma_R$ for a simplicial decomposition $K$ of $X$. It is easy to see that if $M$ is a compact PL $\mathbf{Q}$-manifold possibly with boundary then $M_R$ is a compact definable PL manifold possibly with boundary.

**Theorem**. *The correspondence $M \to M_R$ is a bijection from the PL homeomorphism classes of compact PL $\mathbf{Q}$-manifolds possibly with boundary to the definably PL homeomorphism classes of compact definable PL manifolds possibly with boundary.*

Thus we regard a compact definable PL manifolds possibly with boundary as a compact PL ($\mathbf{Q}$ or) $\mathbf{R}$-manifold possibly with boundary. The proof of the theorem is short but requires some elementary knowledge of PL topology and model theory [8].

## 5. DEFINABLE $C^1$ MANIFOLDS

We will reduce problems on definable $C^1$ manifolds to the $\mathbf{R}$-case. There are many o-minimal structures over fixed $R$. A PL manifold is definable if and only if it is a finite union of open simplexes. Hence definability of a PL manifold does not depend on the choice of an o-minimal structure. However this is not the case for a $C^1$ manifold. Indeed there is a $C^1$ manifold which is definable in an o-minimal structure but not so in another. Hence we need to introduce an o-minimal structure such that a $C^1$ manifold definable in this special structure is definable in any o-minimal structure. That is the *semialgebraic* structure. A *semialgebraic* set is a subset of $R^n$ of the form $\cup\cap\{x \in R^n : f_i(x) *_i 0\}$, where $f_i$ are a finite number of polynomial functions on $R^n$ and $*_i$ means $=$ or $>$. An equivalent definition is that a *semialgebraic* set is a subset of $R^n$ definable in any o-minimal structure. A *semialgebraic* map between semialgebraic sets is a map with semialgebraic graph. It follows that a semialgebraic $C^1$ manifold and a semialgebraic $C^1$ map between semialgebraic $C^1$ manifolds are definable in any o-minimal structure.

By the following theorem we can reduce problems on definable $C^1$ manifolds to the semialgebraic $C^1$ case.

**Theorem** [2]. *A definable $C^1$ manifold is definably $C^1$ diffeomorphic to a semialgebraic $C^1$ manifold.*

Next we will reduce to the $\mathbf{R}$-case. Let $\mathbf{R}_{\mathrm{alg}}$ denote the real algebraic numbers, which is the smallest real closed field. For a polynomial function $f$ on $\mathbf{R}^n_{\mathrm{alg}}$, let $f_R$ denote the polynomial function on $R^n$ naturally extended from $f$. For a semialgebraic set $X = \cup\cap\{x \in \mathbf{R}^n_{\mathrm{alg}} : f_i(x) *_i 0\}$, define $X_R$ to be $\cup\cap\{x \in R^n : f_{iR}(x) *_i 0\}$. Then we easily see that $X$ is a semialgebraic $C^1$ $\mathbf{Q}$-manifold if and only if $X_R$ is a semialgebraic $C^1$ manifold.

**Theorem** [1]. *The correspondence $M \to M_R$ is a bijection from the semialgebraic $C^\infty$ diffeomorphism classes of semialgebraic $C^\infty$ $\mathbf{R}_{alg}$-manifolds to the semialgebraic $C^1$ diffeomorphism classes of semialgebraic $C^1$ $R$-manifolds.*

Thus it suffices to consider semialgebraic $C^\infty$ $\mathbf{R}$- (or $\mathbf{R}_{\mathrm{alg}}$-)manifolds. Compactification of a semialgebraic $C^\infty$ $\mathbf{R}$-manifold is always possible as follows.

**Theorem** [6]. *A non-compact semialgebraic $C^\infty$ $\mathbf{R}$-manifold is semialgebraically $C^\infty$ diffeomorphic to the interior of a compact semialgebraic $\mathbf{R}$-manifold with boundary. Such a compact semialgebraic $\mathbf{R}$-manifold with boundary is unique up to semialgebraically $C^\infty$ diffeomorphisms.*

The proof of the first theorem is the same as the proof of the second. The proof of the second is based on the Morse theory over $R$. The proof of the third uses the Artin-Mazur theorem and the Hironaka desingularization theorem.

Thus the facts shown in section 1 hold for definable $C^1$ manifolds and definable PL manifolds. However, the original proofs of the Cairns-Whitehead theorem and the facts that a PL $\mathbf{R}$-manifold of dimension $\leq 7$ and a compact PL $\mathbf{R}$-manifold admit $C^1$ $\mathbf{R}$-manifold structures are false for non-Archimedean $R$. We can prove them using the above theorems. A typical example of a theorem which holds for $\mathbf{R}$ but not for non-Archimedean $R$ is the simplicial approximation theorem. The simplicial approximation theorem states that any $C^0$ function on an $\mathbf{R}$-polyhedron is approximated by a PL function in the $C^0$ topology. However, if $R$ is non-Archimedean, then the function $R \supset [0, 1] \ni x \to x^2 \in R$ cannot be approximated by a PL function (see [8]).

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